Multidual Quaternions and Higher-Order Kinematics of Lower-Pair Chain

Extended abstract

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Abstract

This paper proposes a new computational method based on vector and quaternionic calculus and the properties of dual and multidual algebra for higher-order acceleration field of lower-pair spatial kinematics chain. A closed form coordinate-free solution is presented, this result being generated by the morphism between the Lie group of the rigid body displacements and the Lie group of the unit multidual quaternions. The solution is put into practice for higher-order kinematics analysis of lower-pair serial chains. A general method for studying the vector field of arbitrary higher-order accelerations is described. The "automatic differentiation" feature of the multidual and hyper-multidual functions is used to obtain the higher-order derivative of a rigid body pose. This is obtained with no need for further differentiation of the body pose with respect to time It is proved that all information regarding the properties of the distribution of higher-order accelerations is contained in the specified unit hyper-multidual quaternion.

Multidual algebra, vectors, quaternions, and higher-order kinematics

Let be $\underline{\mathbb{R}} = \mathbb{R} + \varepsilon_0\mathbb{R}; \varepsilon_0 \neq 0, \varepsilon_0{}^2 = 0$, the set of dual numbers, and $n \in \mathbb{N}$ a natural number. We will introduce the set of hyper-multidual (HMD) numbers by: $\underline{\widehat{\mathbb{R}}} = \underline{\mathbb{R}} + \varepsilon\underline{\mathbb{R}} + \cdots \varepsilon^n\underline{\mathbb{R}};\ \varepsilon \neq 0, \varepsilon^{n+1} = 0$. For n=1, on obtain hyper-dual numbers [5]. The se of multidual (MD) number is introduced by $\widehat{\mathbb{R}} = \mathbb{R} + \varepsilon\mathbb{R} + \cdots + \varepsilon^n\mathbb{R}; \varepsilon \neq 0, \varepsilon^{n+1} = 0$. It can be easily proved that the set $\underline{\widehat{\mathbb{R}}}$ with the addition operation and multiplication is a commutative ring with unit. An element from $\underline{\widehat{\mathbb{R}}}$ is either invertible or zero divisor [6]. The linear $\mathbb{R}$-algebra $\underline{\widehat{\mathbb{R}}}$ is the direct product of dual algebra $\underline{\mathbb{R}}$ and multidual algebra $\widehat{\mathbb{R}}$. $\underline{\widehat{\mathbb{R}}}$ has a structure of 2(n+1) −dimensional associative, commutative, and unitary generalized Clifford Algebra, and $\underline{\widehat{\mathbb{R}}} = \widehat{\mathbb{R}} + \varepsilon_0\widehat{\mathbb{R}}; \varepsilon_0 \neq 0, \varepsilon_0{}^2 = 0$. $\widehat{\mathbb{R}}$ is subalgebra of $\underline{\widehat{\mathbb{R}}}$ of dimension n+1. MD and HMD vectors and tensors was studied in preview paper [3], [4], [5]. An HMD quaternion can be defined as an associated pair of an HMD scalar quantity and a free HMD vector:

$$\underline{\widehat{\mathbf{q}}} = \left(\underline{\widehat{\mathrm{q}}}, \underline{\vec{\widehat{\mathbf{q}}}}\right), \underline{\widehat{\mathrm{q}}} \in \underline{\widehat{\mathbb{R}}}, \underline{\vec{\widehat{\mathbf{q}}}} \in \underline{\widehat{\mathbf{V}}}_3 \tag{1}$$

The set of HMD quaternions will be denoted $\underline{\widehat{\mathbf{Q}}}$ and is a $\underline{\widehat{\mathbb{R}}}$-module of rank 4, if HMD quaternion addition and multiplication with dual numbers are considered. Also, any dual quaternion can be written as $\underline{\widehat{\mathbf{q}}} = \underline{\widehat{\mathrm{q}}} + \underline{\vec{\widehat{\mathbf{q}}}}$, where $\underline{\widehat{\mathrm{q}}} \triangleq \left(\underline{\widehat{\mathrm{q}}}, \underline{\mathbf{0}}\right)$ and $\underline{\vec{\widehat{\mathbf{q}}}}, \triangleq \left(\underline{0}, \underline{\vec{\widehat{\mathbf{q}}}},\right)$, or $\underline{\widehat{\mathbf{q}}} = \widehat{\mathbf{q}} + \varepsilon_0\widehat{\mathbf{q}}_0$, where $\widehat{\mathbf{q}}, \widehat{\mathbf{q}}_0$ are MD quaternions.

Let $\widehat{\mathbb{U}}$ denote the set of MD unit quaternions and $\underline{\widehat{\mathbb{U}}}$ denotes the set of HDM unit quaternions. Also, a HDM number $\underline{\alpha}$ and a unit HDM vector $\underline{\widehat{\mathbf{u}}}$ exist so that:

$$\underline{\widehat{\mathbf{q}}} = \cos\frac{\underline{\widehat{\alpha}}}{2} + \underline{\widehat{\mathbf{u}}}sin\frac{\underline{\widehat{\alpha}}}{2} = \exp\left(\frac{\underline{\widehat{\alpha}}}{2}\underline{\widehat{\mathbf{u}}}\right), \tag{1}$$

where $\underline{\widehat{\alpha}}$ and $\underline{\widehat{\mathbf{u}}}$ are the natural HDM invariants of the rigid body displacement.

The n-th order acceleration of a point of the rigid body given by the position vector $\mathbf{r}$, denotes $\mathbf{a}_{\mathbf{r}}^{[n]}$, can be computed with the following relation [4]:

$$\mathbf{a}_{\mathbf{r}}^{[n]} = \mathbf{a}_n + \mathbf{\Phi}_n \mathbf{r}; n \in \mathbb{N}. \tag{3}$$

Considering the rigid motion parametrized by the dual quaternion function: $\underline{\mathbf{q}} = \underline{\mathbf{q}}(t) \in \underline{\mathbf{U}}, \forall t \in I \subseteq \mathbb{R}$. Where $\underline{\breve{\boldsymbol{\varphi}}} = \underline{\breve{\mathbf{q}}}\,\underline{\mathbf{q}}^*$, $\underline{\mathbf{q}} = \exp(\frac{1}{2}\underline{\alpha}\,\underline{\mathbf{u}})=\cos\frac{1}{2}\underline{\alpha} + \underline{\mathbf{u}}\sin\frac{1}{2}\underline{\alpha}$, $\underline{\breve{\mathbf{q}}} = e^{\varepsilon \mathrm{D}}\underline{\mathbf{q}} = \cos\frac{1}{2}\underline{\breve{\alpha}} + \underline{\breve{\mathbf{u}}}\sin\frac{1}{2}\underline{\breve{\alpha}}$. In the case of helical rigid body motion ($\underline{\breve{\mathbf{u}}} = \underline{\mathbf{u}}$) :

$$\underline{\breve{\boldsymbol{\varphi}}} = \underline{\breve{\mathbf{q}}}\,\underline{\mathbf{q}}^* = \exp(\tfrac{1}{2}\Delta\underline{\breve{\alpha}}\,\underline{\mathbf{u}}) = \cos(\tfrac{1}{2}\Delta\underline{\breve{\alpha}}) + \underline{\mathbf{u}}\sin(\tfrac{1}{2}\Delta\underline{\breve{\alpha}})$$

Consider a spatial kinematic chain of the rigid bodies $C_k, k = \overline{0,m}$ .The relative motion of the rigid body C_k with respect to reference frame attached to C_{k-1} is described by the orthogonal dual unit quaternion ${}^{k-1}\underline{\mathbf{q}}_k$, $k = \overline{1,m}$.The relative motion properties of the terminal body C_m with respect to reference frame attached to C_0 are described by the unit dual quaternion [2], [7]. If unit dual vectors ${}^{k-1}\underline{\mathbf{u}}_k$=const , $k = \overline{1,m}$, the spatial kinematic chain is named general mC manipulator. The following theorem can be proved:

Theorem 3 *The vector fields of higher-order acceleration on terminal body of general mC manipulator given by the kinematic mapping (77) it results from HMD unit dual quaternion*:

$$\underline{\breve{\boldsymbol{\varphi}}} = \exp\left[\frac{1}{2}\underline{\mathbf{u}}_1\Delta\underline{\breve{\alpha}}_1\right]\exp\left[\frac{1}{2}\underline{\mathbf{u}}_2\Delta\underline{\breve{\alpha}}_2\right]\dots\exp\left[\frac{1}{2}\underline{\mathbf{u}}_m\Delta\underline{\breve{\alpha}}_m\right], \tag{4}$$

where $\underline{\mathbf{u}}_1 = {}^0\underline{\mathbf{u}}_1$, *and*:

$$\underline{\mathbf{u}}_k = \mathrm{Ad}_{\,{}^0\underline{\mathbf{q}}_1\,{}^1\underline{\mathbf{q}}_2\dots{}^{k-2}\underline{\mathbf{q}}_{k-1}}({}^{k-1}\underline{\mathbf{u}}_k), k = \overline{2,m} \tag{5}$$

are unit dual vectors corresponding to screw joint k, *and* $\Delta\underline{\breve{\alpha}}_k$ *denote the multidual part of HMD variable* $\underline{\breve{\alpha}}_k$, $k = \overline{1,m}$.

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